Basic concepts of radio astronomy and radiative transfer

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1 Radiation

Radiation is both a wave phenomenon and a particle phenomenon. The speed of light is $c = 2.998 \times 10^8$ m s^{-1}.

- **Wave phenomenon** — We will use λ to refer to wavelength and ν to refer to frequency of a wave. They are related by $c = \lambda\nu$.
- **Particle phenomenon** — The energy of a photon is given by $E = h\nu = hc/\lambda$, and its momentum is given by $p = E/c$. $h = 6.626 \times 10^{-34}$ J s is Planck's constant.

2 Specific intensity

Specific intensity is denoted by I_ν. It is defined as

$$I_\nu = \frac{\mathrm{d}E}{\mathrm{d}A\,\mathrm{d}t\,\mathrm{d}\nu\,\mathrm{d}\Omega} \tag{1}$$

Specific intensity is a measure of "brightness" with units J m^{-2} s^{-1} Hz^{-1} sr^{-1}. It is equal to the energy dE crossing a differential area dA perpendicular to the ray in a time dt in the frequency range ν to $\nu + \mathrm{d}\nu$ in a differential solid angle dΩ. It is a function of position $\vec{r}$, time t, and direction $\vec{\Omega}$, but it does *not* encode information about polarization or reference frame. Specific intensity is intrinsic to the radiation field but is not measured directly — it is inferred. Note that sr is the abbreviation for steradian, a measure of "solid angle" (see Figure 1).

2.1 Moments of specific intensity

If we integrate I_ν over all frequencies, we obtain the frequency-integrated intensity, also called the integrated intensity or bolometric intensity.

$$I \equiv \int_0^\infty I_\nu\,\mathrm{d}\nu \tag{2}$$

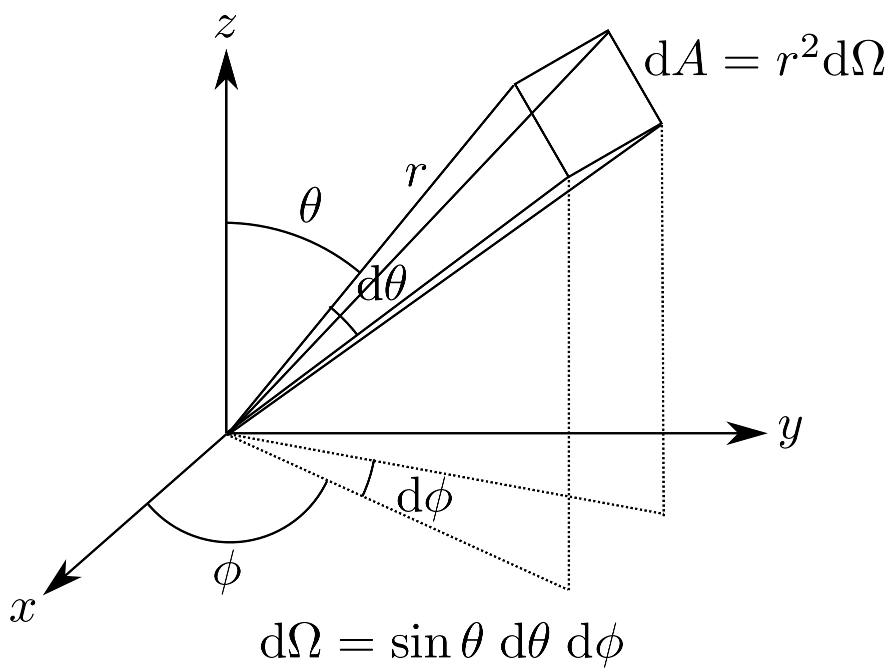

Figure 1: Illustration of a solid angle.

The average intensity over all directions is

$$J_\nu \equiv \frac{\int\limits_\Omega I_\nu \, \mathrm{d}\Omega}{\int\limits_\Omega \mathrm{d}\Omega} \tag{3}$$

$$\equiv \frac{1}{4\pi \text{ sr}} \int\limits_\Omega I_\nu \, \mathrm{d}\Omega, \tag{4}$$

and the average bolometric intensity is given by

$$J \equiv \int\limits_0^\infty J_\nu \, \mathrm{d}\nu. \tag{5}$$

J has units of J cm^{-2} s^{-1} sr^{-1}. The energy density is given by

$$U \equiv \frac{J}{c} \times (4\pi \text{ sr}) \tag{6}$$

and has units of J cm^{-3}. The spectral energy density

$$U_\nu \equiv \frac{J_\nu}{c} \times (4\pi \text{ sr}) \tag{7}$$

also encodes frequency information.

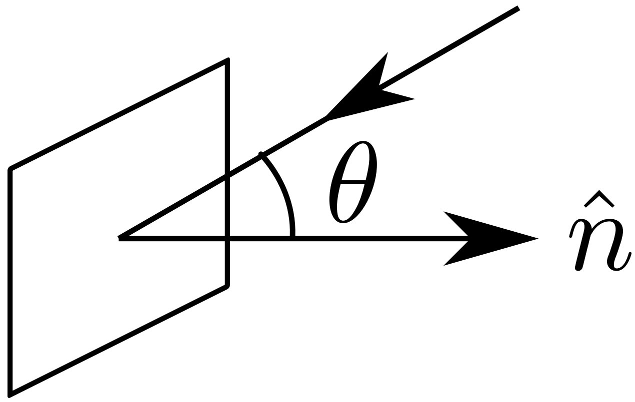

Figure 2: An area element with surface normal $\hat{n}$ and incident ray at angle θ.

2.2 Flux

A ray perpendicular to a surface "sees" the full area; a ray parallel to a surface "sees" no area. Therefore, to find the flux through the surface, the integral must contain a $\cos\theta$ term, where θ is the angle between the ray and the surface normal (see Figure 2). Flux in radio astronomy is commonly written as S_ν. We find the flux by integrating out the solid angle from specific intensity:

$$S_\nu = \int_\Omega I_\nu \cos\theta \, \mathrm{d}\Omega \tag{8}$$

The bolometric flux is defined as

$$S \equiv \int_0^\infty S_\nu \, \mathrm{d}\nu. \tag{9}$$

Suppose the source subtends a small angle θ_0 on the sky, as observed by us. Then we can write Equation 8 as

$$S_\nu = \int_0^{2\pi} \int_0^{\theta_0/2} I_\nu \sin\theta \cos\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \tag{10}$$

$$= 2\pi \int I_\nu \sin\theta \cos\theta \, \mathrm{d}\theta \tag{11}$$

$$\approx 2\pi I_\nu \int \sin\theta \cos\theta \, \mathrm{d}\theta \tag{12}$$

$$= \pi I_\nu \sin^2\left(\frac{\theta_0}{2}\right) \tag{13}$$

$$\approx I_\nu \times \frac{\pi\theta_0^2}{4}, \tag{14}$$

where we apply the small-angle approximation at the last step. We also could have written $S_\nu \approx I_\nu \Omega_0$.

2.3 Pressure

Think of radiation as a gas of photons; each photon carries a momentum. Then the pressure due to radiation is given by

$$P_\nu \equiv \frac{1}{c} \int_\Omega I_\nu \cos^2\theta \, \mathrm{d}\Omega. \tag{15}$$

One factor of $\cos\theta$ in this integral is accounted for by the direction of the photon; the second is related to the direction of the momentum. P_ν has the same units as U_ν, but note that it is a *very* different quantity!

2.4 Intensity as a constant

In a vacuum, intensity is a constant. That means that radiation from a light turned on in Los Angeles is the same by the time it gets to Boston! A sketch of the proof follows.

Consider a collection of all rays that pass through both $\mathrm{d}A_1$ and $\mathrm{d}A_2$, separated by a distance R, as in Figure 3. In a time $\mathrm{d}t$, the energy through $\mathrm{d}A_1$ is

$$\mathrm{d}E_1 = I_{\nu,1} \, \mathrm{d}A_1 \, \mathrm{d}t \, \mathrm{d}\Omega_1 \, \mathrm{d}\nu, \tag{16}$$

and, similarly, the energy through $\mathrm{d}A_2$ is

$$\mathrm{d}E_2 = I_{\nu,2} \, \mathrm{d}A_2 \, \mathrm{d}t \, \mathrm{d}\Omega_2 \, \mathrm{d}\nu. \tag{17}$$

By construction, $\mathrm{d}E_1 = \mathrm{d}E_2$, and we can also write

$$\mathrm{d}\Omega_1 = \frac{\mathrm{d}A_2}{R^2} \tag{18}$$

$$\mathrm{d}\Omega_2 = \frac{\mathrm{d}A_1}{R^2} \tag{19}$$

from the definition of solid angle. Substituting, we find $I_{\nu,1} = I_{\nu,2}$.

2.5 Example of a simple radiation field

Consider the simple field

$$I(\theta) = I_0 \left(1 + \epsilon \cos\theta\right). \tag{20}$$

For this field, we find

$$J = \frac{I_0}{4\pi} \int_0^{2\pi} \int_0^{\pi} (1 + \epsilon \cos\theta) \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi = I_0 \tag{21}$$

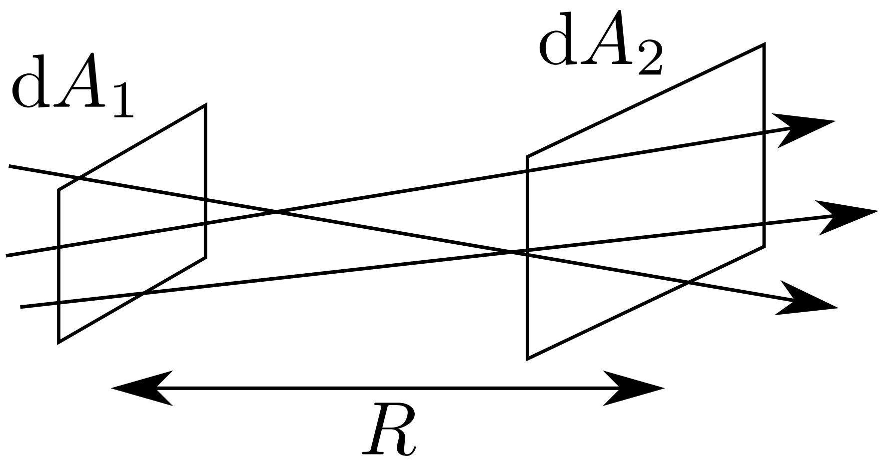

Figure 3: Geometry for the proof that specific intensity is constant along a ray in a vacuum.

$$U = \frac{4\pi}{c} J = \frac{4\pi}{c} I_0 \tag{22}$$

$$S = I_0 \int_0^{2\pi} \int_0^{\pi} (1 + \epsilon \cos\theta) \cos\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi = \frac{4\pi}{3} \epsilon I_0 \tag{23}$$

$$P = \frac{I_0}{c} \int_0^{2\pi} \int_0^{\pi} (1 + \epsilon \cos\theta) \cos^2\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi = \frac{1}{3} U. \tag{24}$$

S characterizes the asymmetry of the radiation, hence the factor of ϵ. $P = \frac{1}{3}U$ is true for this problem (and many others) but is *not* necessarily true for every radiation field.

3 Blackbody radiation

The specific intensity of a blackbody emitter is given by

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1} \tag{25}$$

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1} \tag{26}$$

An interesting property of the blackbody spectrum is that curves at different temperatures will never overlap; increasing the temperature will increase the brightness at all frequencies.

By equating the derivative with respect to ν (or λ) with zero, it is possible to numerically derive the wavelength at which the curve peaks. The relation is not the same for both B_ν and B_λ! The Wien displacement law is

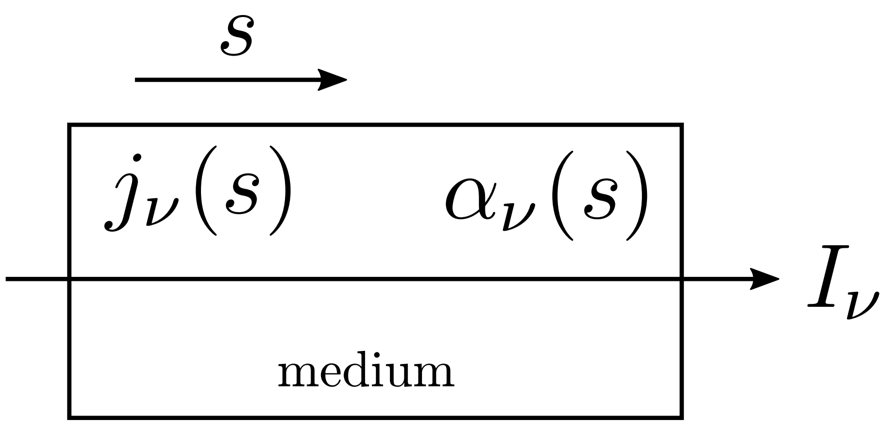

Figure 4: An absorbing and emitting medium, showing the ray coordinate s.

$$\lambda_{\rm max} T = 0.51 \text{ cm K}, \text{ for } B_\nu \tag{27}$$

$$= 0.29 \text{ cm K}, \text{ for } B_\lambda. \tag{28}$$

The first of these equations is the most useful for this class.

When $h\nu \ll kT$, we are in the Rayleigh-Jeans "tail" of the blackbody curve, where we can approximate

$$B_\nu \approx \frac{2\nu^2}{c^2} kT. \tag{29}$$

It is important to remember the regime in which this equation applies, however! It is *not* an appropriate approximation if $h\nu \gg kT$.

With this new approximation for the specific intensity, we can now say

$$S_\nu = I_\nu \Omega = \frac{2\nu^2}{c^2} kT\Omega, \ h\nu \ll kT. \tag{30}$$

For a blackbody, we see that $S_\nu \propto \nu^2$, but not everything is a blackbody. For example, synchrotron radiation has $S_\nu \propto \nu^{-0.7}$.

4 Radiative transfer equation

Let s denote distance along a ray of radiation. We have already shown that $\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = 0$ in a vacuum. Now we consider emitting and absorbing media (see Figure 4).

4.1 Emitting medium

We define

$$j_\nu = \frac{\mathrm{d}E}{\mathrm{d}V\,\mathrm{d}t\,\mathrm{d}\nu\,\mathrm{d}\Omega} \tag{31}$$

as the monochromatic emission coefficient. It has units of J cm^{-3} s^{-1} Hz^{-1} sr^{-1}. In the absence of absorption, the radiative transfer equation is

$$\mathrm{d}I_\nu = j_\nu \,\mathrm{d}s. \tag{32}$$

That is, we are essentially "adding" intensity along the ray, in proportion to the emission coefficient.

4.2 Absorbing medium

Again, let s denote the distance along a ray. The monchromatic absorption coefficient is α_ν, which we can interpret as the inverse length required to reduce the intensity by an e-fold; it is equal to $n\sigma$, where n is the number density of absorbing particles and σ is their cross-sectional area. α_ν has units of cm^{-1}. In the absence of emission, the radiative transfer equation is

$$\mathrm{d}I_\nu = -\alpha_\nu I_\nu \,\mathrm{d}s. \tag{33}$$

This time, we are removing intensity along the ray, in proportion to *both* the absorption coefficient and the current value of I_ν. This makes intuitive sense, because we cannot remove more intensity than we currently have.

Another useful quantity related to absorption is the opacity $\kappa_\nu = \alpha_\nu/\rho$, where ρ is the mass density.

4.3 Emitting and absorbing medium

Putting the previous results together, the full radiative transfer equation is

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = j_\nu - \alpha_\nu I_\nu. \tag{34}$$

5 Temperature as a scale for measurement

The power we observe from a source is $P = kT\Delta\nu$, where $\Delta\nu$ is the bandwidth. We generally characterize radio noise by temperature, since k and $\Delta\nu$ are constant. Some important temperatures you will see in equations in this class are T_{rx}, the receiver temperature, and T_B, the brightness temperature. The brightness temperature is the temperature at which $S_\nu = B_\nu(T)\,\Omega$. Note that this is *not* a physical temperature! It is just a convenient parameterization. For example, pulsars and masers have $T_B \gg 10^9$ K.

6 Statistics

6.1 Expected value and probability

Consider some function f of a random variable x with pdf (probability density function) p. By definition,

$$\int_{-\infty}^{\infty} p(x)\,\mathrm{d}x = 1. \tag{35}$$

The expected value of f is

$$\langle f \rangle = \int f(x)\,p(x)\,\mathrm{d}x. \tag{36}$$

As a simple example, take $f(x) = x$. Then we can compute the expected value of x as

$$\langle x \rangle = \int xp(x)\,\mathrm{d}x. \tag{37}$$

We can also compute, say, the probability of x falling in some range x_1 to x_2 by

$$P(x_1 < x < x_2) = \int_{x_1}^{x_2} p(x)\,\mathrm{d}x. \tag{38}$$

6.2 Multiple variables

For independent variables x, y with $z = x + y$, we have

$$p(z) = p(x) \otimes p(x) \tag{39}$$

$$\mu_z = \mu_x + \mu_y \tag{40}$$

$$\sigma_z^2 = \sigma_x^2 + \sigma_y^2, \tag{41}$$

where the last equation assumes Gaussian noise.

6.3 Gaussian distribution

The Gaussian, or normal, distribution is one of the most important in statistics. It is defined by

$$G(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)}{2\sigma^2}\right] \tag{42}$$

where μ is the mean and σ is the standard deviation. We can easily show that $\langle x \rangle = \mu$ and $\langle (x-\mu)^2 \rangle = \sigma^2$; this second quantity is called the variance.

The standard deviation defines the "width" of the normal distribution, but we could just as easily use the full-width half-max (FWHM), which is defined as the width of the distribution when it is at half its maximum value. For the Gaussian, FWHM $\approx 2.355\sigma$.

6.4 Central Limit Theorem

The Central Limit Theorem states that independent and identically distributed variables $a + b + c + \cdots$ will be normally distributed. See Figure 5 for an example.

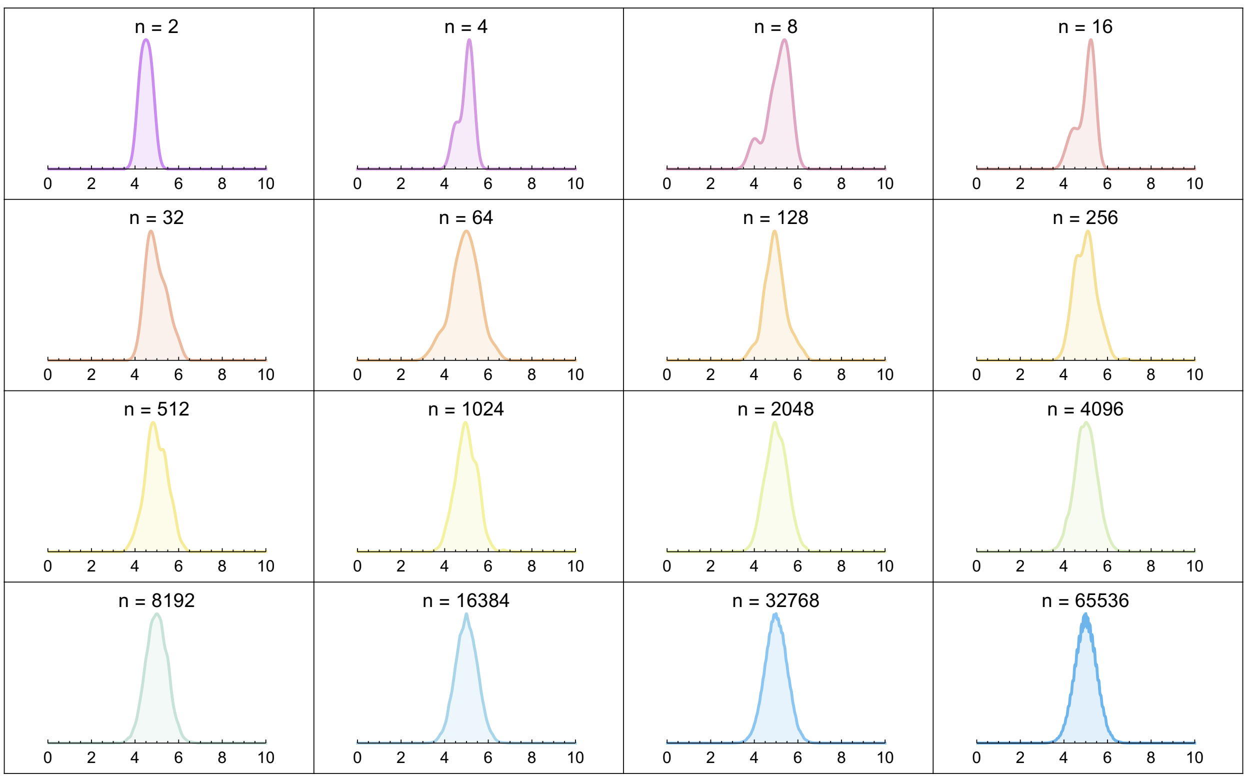

Figure 5: Illustration of the Central Limit Theorem. We take the arithmetic mean of n samples from a binomial distribution $B(10, 0.5)$ and plot the smoothed histogram for increasing n. Visually, this distribution converges to a Gaussian with $\mu = 5$.

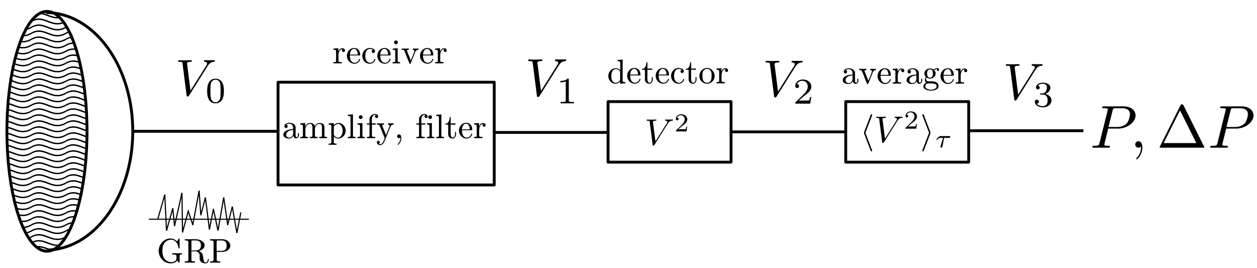

Figure 6: Diagram of antenna, receiver, and detector.

6.5 Photon statistics

In optical astronomy, photons are "rare" events, so we must use Poisson statistics to model them. The Poisson distribution is a discrete distribution defined by

$$P(k) = \frac{\lambda^k}{k!} e^{-\lambda} \tag{43}$$

where λ is the average number of photons per time interval (*not* wavelength) and k is the number of observed photons. Luckily, this isn't optical astronomy! We are not counting individual photons in radio astronomy. The signal we observe is instead a Gaussian random process (GRP).

7 Antennas

Fundamental properties of an antenna are its effective collecting area (related to the geometric area by an efficiency factor $\eta < 1$ by $A_{\text{eff}} = \eta A_{\text{geom}}$), beam angle, and beam shape. The beam angle and solid angles are, approximately,

$$\theta_B \approx \frac{\lambda}{D} \tag{44}$$

$$\Omega_B \approx \frac{\pi \theta_B^2}{4}. \tag{45}$$

From the power equation in the previous section, we can also write $P = \frac{1}{2} S_\nu A_{\text{eff}} \Delta\nu$, where A_{eff} is the effective collecting area and the factor of $1/2$ comes from the fact that the antenna only collects power that is matched to its polarization.

8 Receivers

A receiver (see Figure 6) is characterized by its temperature T_{rx}, gain G, and bandwidth $\Delta\nu$. To whatever signal we detect with a receiver, we add noise $G k T_{\text{rx}} \Delta\nu$; quantum theory dictates that $T_{\text{rx}} > h\nu/k$.

- By construction, V_0 is a Gaussian random variable (GRV). $\langle V_0 \rangle = 0$ and $\langle V_0^2 \rangle = k T_A \Delta\nu$.

- V_1 is also a GRV, since it depends linearly on V_0. $\langle V_1 \rangle = 0$ and $\langle V_1^2 \rangle = Gk\,(T_A + T_{\rm rx})\,\Delta\nu$.
- $V_2 = V_1^2$ is *not* a GRV. $\langle V_2 \rangle = \langle V_1^2 \rangle$ and $\langle V_2^2 \rangle = \langle V_1^4 \rangle = 3\langle V_1^2 \rangle^2 = 3\left[Gk\,(T_A + T_{\rm rx})\,\Delta\nu\right]^2$. This result depends on Isserlis' Theorem:

$$\langle x^n \rangle = \begin{cases} 0, & n \text{ odd} \\ (n-1)\,\sigma^n, & n \text{ even} \end{cases} \tag{46}$$

  The variance is $\sigma_2^2 = \langle V_2^2 \rangle - \langle V_2 \rangle^2 = 2\left[Gk\,(T_A + T_{\rm rx})\,\Delta\nu\right]^2$.
- $V_3 = \langle V_2 \rangle_\tau$, so $\langle V_3 \rangle = \langle V_2 \rangle = Gk\,(T_A + T_{\rm rx})\,\Delta\nu$ and $\sigma_3^2 = \sigma_2^2/\left(2\tau\Delta\nu\right)$.